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OPTIMAL REPLACEMENT: AN EXTENSION

TO CONSUMER DURABLES

by

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and

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Optimal Replacement: An Extension to Consumer Durables*

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I. INTRODUCTION

The problem of determining the optimal procedure for the replacement of capital or military equipment has been of constant concern to industrial firms and military organizations. Although numerous studies exist in this area,[1] having been conceived and developed in conjunction with investment and inventory theory, the study of optimal replacement has never been effectively extended to the area of consumer durables.[2]

The primary objective of this paper is to examine such a problem within an inter-temporal utility maximization framework,[3] in which a consumer has a finite, fixed time horizon (T)[4] and derives satisfaction from the consumption of both durable and non-durable goods. In particular, we choose housing as the consumer durable upon which we base our discussion. The following section introduces a fairly detailed exposition of the model, in which replacement occurs once. Section 3 derives and characterizes the optimal consumption sequence. The well-known Fisherian result is observed, i.e., that the optimal consumption sequence is increasing (decreasing) provided the ratio of interest rate to the subjective time preference is larger (smaller) than unity. Section 4 considers the problem of comparative dynamics, i.e., changing parameters such as the price of a new house or the market rate of interest and observing the corresponding changes in the consumption sequence. It is noted that the change in the price of a new house shifts the entire consumption sequence either upward or downward depending upon what happens to the final stock of wealth as the price changes. The ambiguity of the results presented arises because of the presence of "wealth effect" which correspond closely with the comparable "income effects" of static analysis. The changes in the market rate of interest have a more complex effect upon the consumption sequence than the price change. The effect of price changes and interest rate changes upon the optimal replacement time is also examined.

II. THE MODEL

Consider a consumer who at present owns a house, but has the option to replace it some time in the future. He would like to determine his optimal sequence of saving and consumption, as well as the optimal point in the future to replace the house, given a finite, fixed time horizon.[5] He derives utility from the house in which he lives, from the rest of goods and services he consumes, and from the final stock of wealth. Thus the consumer's discounted present value of utility consists of the sum of his discounted instantaneous utility and the discounted utility of his final stock of wealth. We assume that the use value of a house can be approximated by its size and quality, as well as environmental considerations such as the nature of the school district, tax rate, zoning, neighborhood, etc. We also assume that this use value declines over the years due to general wear and tear and obsolescence.

We will use the following notation:

y_t	-	(Exogenously given) income at time t.
C_t	-	consumption of goods and services, excluding housing, at time t.
h_t	-	the use value of house at time t.
h^0	-	use value of the old house at the time of its purchase $t_0^* \le 0$
h^1	-	use value of the new house at the time of its purchase t^*
ρ	-	depreciation rate of a house.
$U(C_t, h_t)$	-	utility at time t.
μ	-	the consumer's rate of time preference.
W_T	-	final stock of wealth.
$V(W_T)$	-	utility of the final stock of wealth.

s - T-component vector of periodic saving, s_t.

t^* - the replacement time, i.e., at time t^* he sells his old house and buys a new one.

P - the price of a new house.

r - the market rate of interest.

d_t - the individual's mortgage payment at time t.

Then the discounted present value of utility is given by

$$\Psi(s,t^*,P) = \sum_{t=1}^{T} U(C_t,h_t)(1+\mu)^{1-t} + V(W_T)(1+\mu)^{1-T} \tag{1.1}$$

where

$$h_t = \begin{cases} h^0 e^{-\rho(t-t_0^*)} & t = 1,\ldots,t^* - 1 \\ h^1 e^{-\rho(t-t^*)} & t = t^*,\ldots,T \end{cases} \tag{1.1a}$$

We assume that the size of one's mortgage payment d_t, is non-negative, known with certainty and equal to $\bar{d}$ as long as the old house is occupied[6] (i.e., for $t = 1,\ldots,t^* - 1$), and equal to $f(P)$ thereafter (i.e., for $t = t^*,\ldots,T$), with $f'(P) > 0$.

If we analogously define m_t as the exogenously given maintenance cost of the occupied house at time t, with $m^i(\tau)$ being the maintenance cost of the i^{th} (i.e., either old $(i=0)$ or new $(i=1)$) house of age τ, then we may write

$$C_t = y_t - s_t - d_t - m_t \qquad t = 1,\ldots,T \tag{1.2}$$

$$d_t = \begin{cases} \bar{d} & t = 1,\ldots,t^* \\ f(P) & t = t^*+1,\ldots,T \end{cases} \tag{1.2a}$$

$$m_t = \begin{cases} m^0(t-t_0^*) & t = 1,\ldots,t^*-1 \\ m^1(t-t^*) & t = t^*,\ldots,T \end{cases} \tag{1.2b}$$

Further, note that one's stock of wealth at time t, W_t, is the sum of one's stock of savings, s_t and the equity in one's home. Equity at time t is the difference between the market value of the house, H_t and the outstanding debt on it, D_t.

In particular, we are interested in the equity value of his old house at time t^*, $E^0_{t^*}$.

$$E^0_{t*} = H^0_{t*} - D^0_{t*} \tag{1.3a}$$

We may write the market value of the old house at the time of its sale as an increasing function of both its use value and the price of new housing. Thus,

$$H^0_{t*} = H(h^0_{t*}, P) \tag{1.3b}$$

Next, writing the outstanding debt on his old house in terms of the original debt level and the rate of interest we have:

$$D^0_{t*} = (1+r)^{t*} D^0_0 - \bar{d} \sum_{t=1}^{t*} (1+r)^{t-1} \geq 0 \quad 7 \tag{1.3c}$$

Thus, letting $\emptyset(P)$ be the downpayment on the new house such that we have:

$$S_t = \begin{cases} s_t + (1+r)S_{t-1} & t \neq t^*,\ t = 1,\ldots,T \\ s_t + (1+r)S_{t-1} + \{E^0_{t^*} - \emptyset(P)\} & t = t^* \end{cases} \tag{1.3}$$

$$\frac{d\emptyset}{dP} = \emptyset'(P) \in (0,1),\ \emptyset(0) = 0.$$

Thus, given the initial stocks of saving S_0 and debt D^0_0, we can express S_t as follows:

$$S_t = \begin{cases} (1+r)^t S_0 + \sum_{k=1}^{t} (1+r)^{t-k} s_k & t = 1,\ldots,t^*-1 \\ (1+r)^t S_0 + \sum_{k=1}^{t} (1+r)^{t-k} s_k + (1+r)^{t-t^*} \{E^0_{t^*} - \emptyset(P)\} & t = t^*,\ldots,T \end{cases} \tag{1.3^1}$$

Next, writing the market value of the new house at the end of the horizon as a fraction of the cost per use value multiplied by the total use value at time T we have

$$H^1_T = \gamma(P/h^1)h^1_T,\ \gamma \in (0,1) \tag{1.4a}$$

And, since the outstanding debt on the new house at time T may be expressed as

$$D^1_T = (1+r)^{T-t^*}\{P-\emptyset(P)\} - f(P) \sum_{t=t^*+1}^{T} (1+r)^{t-t^*-1} \geq 0 \tag{1.4b}$$

given $\bar{t}$,[8] the time period allowed to complete the debt payment,

we may write

$$D_T^1 = \{P-\emptyset(P)\}\ B(t*) \qquad (1.4b^1)$$

where

$$B(t*) = \frac{1 - (1+r)^{T-(t*+\bar{t})}}{1 - (1+r)^{-\bar{t}}} \in [0,1]$$

$$\frac{dB(t^*)}{dt^*} > 0 \ .$$

Thus, since final wealth equals the stock of savings plus equity we have:

$$W_T = \begin{cases} (1+r)^T S_0 + \sum_{k=1}^{T} (1+r)^{T-k} s_k + (1+r)^{T-t^*} \{E_{t^*}^0 - \emptyset(P)\} & \\ \quad + \gamma(P/h^1)h_T^1 - \{P - \emptyset(P)\}\ B(t*) & \text{for } t* \leq T \\ (1+r)^T S_0 + \sum_{k=1}^{T} (1+r)^{T-k} s_k + H(h_T^0,P) & \\ \quad - (1+r)^T D_0^0 + \bar{d} \sum_{t=1}^{T} (1+r)^{t-1} & \text{for } t^* > T \end{cases} \qquad (1.4)$$

Thus, we wish to maximize (1.1) above with respect to s and t^* for the given price of the new house, subject to (1.1a) through (1.4) above, plus (1.5) through (1.8c) below which are the usual boundry and convexity constraints.

$$C_t \geq 0 \quad \forall\ t \qquad (1.5)$$

(9)

$$S_{t^*} \geq 0 \qquad (1.6)$$

$$W_T \geq 0 \qquad (1.7)$$

$$U(C_t,h_t) > 0,\ V(W_T) > 0 \qquad \text{for} \quad C_t,h_t > 0,\ W_T \geq 0 \tag{1.8a}$$

$$\frac{\partial}{\partial C_t} U(C_t,h_t) > 0,\ \frac{\partial}{\partial h_t} U(C_t,h_t) > 0,\ \frac{\partial}{\partial C_t} U(0,h_t) = \infty,$$

$$\frac{d}{dW_T} V(W_T) > 0, \qquad \text{for } C_t,h_t > 0,\ \ W_T \geq 0. \tag{1.8b}$$

$$\frac{\partial^2}{\partial C_t^2} U(C_t,h_t) < 0,\ \frac{d^2}{dW_T^2} V(W_T) < 0 \qquad \text{for } C_t,h_t > 0,\ W_T \geq 0. \tag{1.8c}$$

III. DERIVATION AND CHARACTERIZATION OF THE OPTIMAL CONSUMPTION SEQUENCE

The maximization can be performed in two steps:

(A) Maximize the discounted present value of utility with respect to s_t, $t = 1,\dots,T$ for given P and t^*.

(B) Perform the above operation for every $t^* = 1,\dots,T+1$ for the given P and choose t^* which maximize the discounted present value of utility.

We note that since the characterization of the optimal consumption sequence is invariant under different values of t^*,[10] it suffices to perform step (A) above in order to characterize the optimal consumption sequence.

We form the following Lagrange function:

$$\begin{aligned} L = \sum_{t=1}^{T} U(y_t - s_t - d_t - m_t, h_t)(1+\mu)^{1-t} + (1+\mu)^{1-T} V[(1+r)^T S_0 \\ + \sum_{k=1} (1+r)^{T-k} s_k + (1+r)^{T-t^*}\{E^0_{t^*} - \emptyset(P)\} + \gamma(P/h^1)h^1_T \\ - \{P - \emptyset(P)\}\, B(t^*)] + \alpha s_{t^*} + \beta W_T \end{aligned} \qquad (1.9)$$

where α,β are the Lagrange multipliers associated with $S_{t^*} \geq 0$, $W_T \geq 0$.

$$\frac{\partial U}{\partial C_t} \geq \frac{dV}{dW_T}\left(\frac{1+r}{1+\mu}\right)^{T-t} \quad \text{for } t = 1,2,\dots,T \qquad (1.10)$$

(Equality holds for $S_{t^*}, W_T > 0$)

For the case of an interior maximum, i.e., $S_{t^*}, W_T > 0$, we derive from (1.8) and (1.10) above the following three possibilities depending upon the relationship between housing consumption and the consumption of other goods and services in the utility function.

Case a If their consumption is independent, i.e., $\frac{\partial^2 U}{\partial C_t \partial h_t} = 0$ then the optimal consumption sequence $\{C_t\}_{t=1}^{T}$ is strictly increasing, constant and strictly decreasing for $r > \mu$, $r = \mu$, $r < \mu$, respectively.

Case b If they are non-complements, i.e., $\frac{\partial^2 U}{\partial C_t \partial h_t} \leq 0$ then the optimal consumption sequences $\{C_t\}_{t=1}^{t^*}$, $\{C_t\}_{t=t^*+1}^{T}$ are both strictly increasing and non-decreasing for $r > \mu$, $r = \mu$, respectively.

Case c If they are non-substitutes, i.e., $\frac{\partial^2 U}{\partial C_t \partial h_t} \geq 0$ then the optimal consumption sequences $\{C_t\}_{t=1}^{t^*}$, $\{C_t\}_{t=t^*+1}^{T}$ are both non-increasing and strictly decreasing for $r = \mu$, $r < \mu$, respectively.

We note in all of the three cases the importance of the relative magnitude of the interest rate to the rate of subjective time preference in determining the optimal consumption sequence. In general if the interest rate is greater than the rate of time preference, it is preferable to save more now and earn the interest which can be spent later without undue sacrifice. The converse also holds.

IV. COMPARATIVE DYNAMICS

First consider the effect of a change in the price of the new house, P, on the optimal consumption sequence $\{C_t\}_{t=1}^{T}$ and the optimal replacement timing t^*:

From (1.10) above we derive the following

$$\frac{\partial U/\partial C_j}{\partial U/\partial C_i} = \left(\frac{1+r}{1+\mu}\right)^{i-j} \qquad \text{for } S_{t*}, W_T > 0 \text{ and } i,j=1,\ldots,T. \tag{1.11}$$

Differentiate (1,11) with respect to P to obtain

$$\frac{dC_j/dP}{dC_i/dP} = \left(\frac{\partial^2 U/\partial C_i^2}{\partial^2 U/\partial C_j^2}\right)\left(\frac{1+r}{1+\mu}\right)^{i-j} > 0 \tag{1.12}$$

$$\text{for } S_{t*} > o, W_T > 0 \text{ and } i,j=1,\ldots,T.$$

which states that the price change will <u>shift</u> the whole consumption sequence. Thus if we know what happens to the level of consumption in any one period, we know for the rest of periods. Proposition 1 and its corollary state that if we know what happens to the final stock of wealth as the price changes, we can tell the corresponding changes in the optimal consumption sequence and the present value of utility, Ψ.

<u>Proposition 1</u>

For given $t*$, if $S_{t*}, W_T > 0$, then $dW_T/dP \geq 0$ implies[11].

(i) $dC_t/dP \geq 0 \ \forall t$, (ii) $d\Psi/dP \geq 0$ (and equality holds for $dW_T/dP = 0$).

Proof: (i) To show $dC_t/dP \geq 0 \ \forall \ t=1,2,\ldots,T$.

$dW_T/dP \geq 0$ implies $\frac{d}{dP} V'(W_T) \leq 0$ by strict concavity of $V(W_T)$.

From (1,10) $\frac{\partial U}{\partial C_t} = V'\left(\frac{1+r}{1+\mu}\right)^{T-t}$ $\forall t$, thus $\frac{d}{dP}\left(\frac{\partial U}{\partial C_t}\right) \leq 0$ $\forall t$.

However $\frac{dh_t}{dP} = 0$ and $\frac{\partial^2 U}{\partial C_t^2} < 0$ imply $\frac{dC_t}{dP} \geq 0$ $\forall t$.

Q.E.D.

(ii) To show $d\Psi/dP \geq 0$

Differentiate (1.1) with respect to P to obtain

$$\frac{d\Psi}{dP} = \sum_{t=1}^{T} \frac{\partial U}{\partial C_t} \frac{dC_t}{dP} (1 + \mu)^{1-t} + \frac{dV}{dW_T} \frac{dW_T}{dP} (1 + \mu)^{1-T} \tag{1.13}$$

By the result above $dC_t/dP \geq 0$ $\forall t$ and by hypothesis $dW_T/dP \geq 0$, we have $d\Psi/dP \geq 0$.

Q.E.D.

<u>Corollary 1</u>

For given t^* assume $S_{t^*}, W_T > 0$, then $dW_T/dP < 0$ implies

(i) $dC_t/dP < 0$ $\forall t$, (ii) $d\Psi/dP < 0$.

In general, it is not possible to find the directional change of the optimal replacement time (t^*) as the price of a new house changes. The change of the replacement time in either direction would have both a positive and negative effect on the utility. However, by positing some additional restrictions on the nature of the utility functions and the end conditions, we obtain the following two propositions with opposite results.

<u>Proposition 2-1</u>

If (1) the optimal final stock of wealth associated with the different prices of a house are the same and positive, and the stock of saving at the time of replacement associated with the different prices are positive; (2) the consumption and housing services are independent items in his utility function; (3) the consumer's time preference equals the market rate of interest; (4) the new house yields higher use-value than the old house, then an increase in the price of a house will not delay replacement, provided that (5) the final stock of wealth does not fall for the given replacement time. The assumptions imply:

$$(1) \qquad W_T(P_1) = W_T(P_2) > 0; \qquad S_{t^*(P_1)}, S_{t^*(P_2)} > 0$$

$$(2) \qquad \frac{\partial^2 U(C_t, h_t)}{\partial C_t \partial h_t} = 0$$

$$(3) \qquad r = \mu$$

$$(4) \qquad h^1 e^{-\rho(T-b+1-t^*(P_1))} \geq h^0 e^{-\rho[t^*(P_1)-t_0^*]}$$

$$(5) \qquad \left.\frac{dW_T}{dP}\right|_{t^* \text{ fixed}} \geq 0$$

then $P_1 < P_2$ implies $t^*(P_1) \geq t^*(P_2)$.

Proof:

(i) To show that the present value of utility would not fall, define $t^*(P_i) \ni \Psi[t^*(P_i),P_i] \geq \Psi[t^*,P_i] \forall t^*$, $i = 1,2$. And note that assumptions (1) and (5), and Proposition 1 yield

$$\Psi[t^*(P_1),P_1] \leq \Psi[t^*(P_1),P_2] .$$

Thus by definition of $t^*(P)$ we obtain:

$$\Psi[t^*(P_1),P_1] \leq \Psi[t^*(P_2),P_2] \tag{1.14}$$

Q.E.D.

(ii) To show $t^*(P_1) \geq t^*(P_2)$, assume the contrary, i.e., $t^*(P_1) < t^*(P_2)$. Assumptions (1), (2) and (3) ensure the optimal consumption sequences $\{C_t^1\}_{t=1}^T$ and $\{C_t^2\}_{t=1}^T$ are identical and constant over time, where C_t^j is the consumption at time t associated with price P_j.

Case 1 Assume $T = t^*(P_2) > t^*(P_1)$, and let h_t^j be the use value of a house at time t associated with P_j. Then $h_t^1 \geq h_t^2$ $\forall$ $t=1,\ldots,T$. Thus

$$\Psi[t^*(P_1),P_1] > \Psi[t^*(P_2),P_2].$$

Case 2 Assume $T > t^*(P_2) = t^*(P_1) + b$ where b is any arbitrary positive integer such that the inequality above is satisfied.

We have

(1) $h_t^1 = h_t^2$ $\quad t = 1,\ldots,t^*(P_1)-1$ (area A in Fig. 1)

(2) $h_t^1 = h_{t+b}^2$ $\quad t = t^*(P_1),\ldots,T-b$ (area B in Fig. 1)

(3) $h_{T-b+1}^1 \geq h_{t^*(P_1)}^2$ by assumption (4), which implies that

$$h^1_{t+T-b-t^*(P_1)+1} \geq h_t^2 \qquad t = t^*(P_1),\ldots,t^*(P_1)+b-1$$

(area C in Fig. 1)

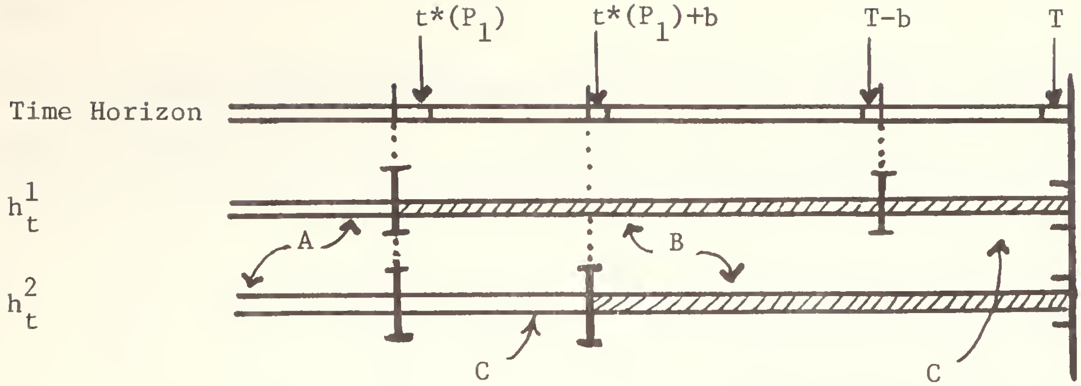

FIGURE 1

(Shaded area corresponds to the use value from the new house)

Thus we obtain

$$\sum_{t=t^*(P_1)}^{T} U(C_t^1,h_t^1)(1+\mu)^{1-t} > \sum_{t=t^*(P_1)}^{t^*(P_1)+b-1} U(C_t^1,h^1_{t+T-b-t^*(P_1)+1})(1+\mu)^{1-t}$$

$$+ \sum_{t=t^*(P_1)+b}^{T} U(C_t^1,h_{t-b}^1)(1+\mu)^{1-t} \geq \sum_{t=t^*(P_1)}^{T} U(C_t^2,h_t^2)(1+\mu)^{1-t}.$$

Hence $\Psi[t^*(P_1),P_1] > \Psi[t^*(P_2),P_2]$, contradicting (1.14) above. Thus we conclude $t^*(P_1) \geq t^*(P_2)$.

Q.E.D.

Proposition 2-2

Under the identical assumptions (1) through (4) in Proposition 2-1, the increase in the price would not hasten the replacement time, provided that (5) the final stock of wealth declines for any given t^*.

The proof is analogous to that for Proposition 2-1, substituting

(5^1) $\left.\frac{dW_T}{dP}\right|_{t^* \text{ fixed}} < 0$ for (5) and using Corollary 1 instead of Proposition 1.

Next we consider the effect of change in the rate of interest on the optimal consumption sequence $\{C_t\}_{t=1}^{T}$ and the optimal replacement timing t^*:

From (1.10) above we derive

$$\frac{\partial U/\partial C_t}{\partial U/\partial C_{t+1}} = \left(\frac{1 + r}{1 + \mu}\right) \text{ for } S_{t^*}, W_T > 0 \text{ and } \forall t = 1,\ldots,T-1. \tag{1.15}$$

Differentiating (1.15) with respect to r we obtain

$$a_1 \frac{dC_1}{dr} < a_1 a_2 \frac{dC_2}{dr} < \cdot\cdot\cdot < \prod_{i=1}^{t} a_i \frac{dC_t}{dr} < \cdot\cdot\cdot < \prod_{i=1}^{T} a_i \frac{dC_T}{dr} \tag{1.16}$$

where

$$a_t = \left(\frac{1 + r}{1 + \mu}\right)\left(\frac{\partial^2 U/\partial C_t^2}{\partial^2 U/\partial C_{t-1}^2}\right) > 0 \qquad t = 2,3,\ldots,T; \quad a_1 = 1.$$

Thus it is clear that if the optimal consumption in the first period rises as the interest rate rises, then the consumption in each following period will also rise. On the other hand if the optimal consumption in the final period falls as the interest rises, then so does the consumption in each preceding period. Thus we obtain Proposition 3 and its Corollary.

Proposition 3.

For given t^*, if $S_{t^*}, W_T > 0$, then $\frac{dC_1}{dr} \geq 0$ implies

(i) $$\frac{dC_t}{dr} > 0 \qquad t = 2,\ldots,T$$

(ii) $$\frac{d\Psi}{dr} > 0 .$$

Proof:

(i) is immediate by (1.16)

(ii) To show $\frac{d\Psi}{dr} > 0$:

Differentiating (1.10) with respect to r yields

$$\frac{\partial^2 U}{\partial C_t^2} \frac{dC_t}{dr} = V'' \frac{dW_T}{dr} \left(\frac{1+r}{1+\mu}\right)^{T-t} + (T-t)\ V' \left(\frac{1+r}{1+\mu}\right)^{T-t-1} \left(\frac{1}{1+\mu}\right) \forall t \quad (1.17)$$

From (i) we have $\frac{dC_t}{dr} > 0 \quad t = 2,\ldots,T$, thus

$$V'' \frac{dW_T}{dr} \left(\frac{1+r}{1+\mu}\right)^{T-t} + (T-t)V'\left(\frac{1+r}{1+\mu}\right)^{T-t-1} \left(\frac{1}{1+\mu}\right) < 0\ ,$$

$$t = 2,\ldots,T \quad (1.18)$$

which implies $\frac{dW_T}{dr} > 0$. Differentiate (1.1) with respect to r to obtain

$$\frac{d\Psi}{dr} = \sum_{t=1}^{T} \frac{\partial U}{\partial C_t} \frac{dC_t}{dr} (1+\mu)^{1-t} + V' \frac{dW_T}{dr} (1+\mu)^{1-T} \quad (1.19)$$

Thus we have $\frac{d\Psi}{dr} > 0$.

Q.E.D.

Corollary 3

For given t^* if $S_{t^*}, W_T > 0$, then $\frac{dC_T}{dr} \leq 0$ implies

(i) $\frac{dC_t}{dr} < 0 \quad t=1,\ldots,T-1$

In addition if the final stock of wealth does not increase as the rate of interest rises, i.e., $\frac{dW_T}{dr} \leq 0$, then

(ii) $\frac{d\Psi}{dr} < 0$. 12

As in the previous section, a change in the final stock of wealth has a significant impact on the change in the consumption sequence and the present value of utility.

Proposition 4.

For given t^* if $S_{t^*}, W_T > 0$, then $\frac{dW_T}{dr} \leq 0$ implies

(i) $\frac{dC_t}{dr} < 0 \qquad t = 1,2,\ldots,T-1 \ ; \quad \frac{dC_T}{dr} \leq 0$,

(ii) $\frac{d\Psi}{dr} < 0.$

Proof.

Recalling (1.17)

$$\frac{dW_T}{dr} \leq 0 \quad \text{implies} \quad \frac{\partial^2 U}{\partial C_t^2} \frac{dC_t}{dr} > 0 \qquad t = 1,\ldots,T-1 \quad \text{and}$$

$$\frac{\partial^2 U}{\partial C_T^2} \frac{dC_T}{dr} \geq 0, \quad \text{hence} \quad \frac{dC_t}{dr} < 0 \qquad t = 1,\ldots,T-1 \quad \text{and} \quad \frac{dC_T}{dr} \leq 0.$$

By the equation (1.19) above, we have $\frac{d\Psi}{dr} < 0.$

Q.E.D.

The next logical step is derivation of the factors involved in the determination of the sign [13] of dC_1/dr and dC_T/dr. Expressing the equation (1.17) explicitly we have,

$$U_{tt} \frac{dC_t}{dr} = a_t \frac{dW_T}{dr} + a_t (T-t) b \tag{1.20}$$

$$= a_t B - a_t \sum_{t=1}^{T} (1 + r)^{T-t} \frac{dC_t}{dr} + a_t (T-t) b \ \forall t$$

where

$$U_{tt} = \frac{\partial^2 U}{\partial C_t^2} < 0, \qquad a_t = V''\left(\frac{1 + r}{1 + \mu}\right)^{T-t} < 0 \tag{1.20a}$$

$$b = V'/_{V''(1+r)} \tag{1.20b}$$

$$B = T(1+r)^{T-1}(S_0-D_0^0) + \sum_{t=1}^{T} (T-t)(1+r)^{T-t-1} s_t \tag{1.20c}$$

$$+ (T-t^*)(1+r)^{T-t^*-1}(H_{t^*}^0) + \bar{d} \sum_{t=1}^{t^*} (T-t^*-1+t)(1+r)^{T-t^*-2+t}$$

$$+ \{1-(1+r)^{-\bar{t}}\}^{-2} \{(t^*+\bar{t}-T)(1+r)^{T-\bar{t}-t^*-1} + (T-t^*)(1+r)^{T-2\bar{t}-t^*-1}$$

$$- \bar{t}(1+r)^{-\bar{t}-1}\} \{P - \phi(P)\} - (T-t^*)(1+r)^{T-t^*-1}\phi(P)$$

$$\frac{dW_T}{dr} = (B - \sum_{t=1}^{T} (1+r)^{T-t} \frac{dC_t}{dr}) \tag{1.20d}$$

Collecting terms and presenting them in a matrix form, we have $Ax = b$, or

$$\begin{bmatrix} U_{11}+a_1(1+r)^{T-1} & a_1(1+r)^{T-2} & \cdots & a_1 \\ a_2(1+r)^{T-1} & U_{22}+a_2(1+r)^{T-2} & \cdots & a_2 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_T(1+r)^{T-1} & a_T(1+r)^{T-2} & \cdots & U_{TT}+a_T \end{bmatrix} \begin{bmatrix} \frac{dC_1}{dr} \\ \frac{dC_2}{dr} \\ \cdot \\ \cdot \\ \cdot \\ \frac{dC_T}{dr} \end{bmatrix} = \begin{bmatrix} a_1(B+(T-1)b) \\ a_2(B+(T-2)b) \\ \cdot \\ \cdot \\ a_{T-1}(B+b) \\ a_T B \end{bmatrix} \tag{1.21}$$

The determinant of A and the typical cofactor of A are given by

$$|A| = \prod_{t=1}^{T} U_{tt} + \sum_{k=1}^{T} a_k(1+r)^{T-k} \prod_{t\neq k} U_{tt} \tag{1.22}$$

$$A_{ij} = -a_j(1+r)^{T-i} \prod_{t\neq i,j} U_{tt} \quad i\neq j, \qquad i,j=1,2,\ldots,T \tag{1.23}$$

$$A_{ii} = \prod_{t\neq i} U_{tt} + \sum_{k\neq i} a_k(1+r)^{T-k} \prod_{t\neq i,k} U_{tt} \qquad i=1,2,\ldots,T$$

Solving for dC_j/dr we have

$$\frac{dC_j}{dr} = \frac{a_j}{|A|}\{B \prod_{t\neq j} U_{tt} + b(T-j) \prod_{t\neq j} U_{tt} + b \sum_{i\neq j} a_i(1+r)^{T-i} \prod_{t\neq i,j} U_{tt}(i-j)\} \tag{1.24}$$

$$j = 1,2,\ldots,T.$$

Thus

$$\frac{dC_1}{dr} = \frac{a_1}{|A|}\{B \prod_{t\neq 1} U_{tt} + b(T-1) \prod_{t\neq 1} U_{tt} + b \sum_{1\neq 1} a_i(1+r)^{T-i} \prod_{t\neq i,1} U_{tt}(i-1)\} \tag{1.25}$$

It is clear that the necessary condition for dC_1/dr to be non-negative is to have $B > 0$. Recalling (1.20c) we note the difficulty in signing B due to the unknown sign pattern of saving (s_t). It can be said, however, that an increase in the initial stock of saving S_0 or an increase in the market value of the old house H_t^0*, or a decrease in the initial debt D_0^0 will have a tendency to increase B and hence consumption. On the other hand the necessary condition for dC_T/dr to be non-positive is to have $B < 0$ as can be seen by choosing $j = T$ in the equation (1.24).

Finally, although the conventional wisdom accepts that an increase in the mortgage rate of interest will cause a decrease in home buying, (a delay in t^* in our terms) we show in the following proposition that there exists a set of market conditions such that an increase in r <u>hastens</u>

replacement. This situation obtains if a purchase is actually made during the time period, if the debt payment period is relatively long, and if the final period outstanding debt as a result of the increase in the rate of interest is less than or equal to that at the original rate. This latter condition may obtain for example, when there exists an effective total debt constraint with respect to the end of the horizon (T) at the original interest rate.

Although certainly not directly applicable in a policy sense because of the non-general-equilibrium nature of the model, this may indicate an area for potentially fruitful policy research in the housing market.

<u>Proposition 5</u>

If (1) $\bar{t}$ is such that $(t^*+\bar{t}-T)\ \ln(1+r) \geq 1$

(2) $D_T^1(r_1) = D_T^2(r_2) = \bar{D}$

then $r_1 < r_2$ implies $t^*(r_1) > t^*(r_2)$ for $\forall\ t^*(r_1) < T$.

Proof:

(i) To show $\left.\frac{dD_T^1}{dr}\right|_{t^*\ \text{fixed}} > 0$:

Differentiate equation $(1.4b^1)$ with respect to r to obtain

$$\frac{dD_T^1}{dr} = (P-\emptyset)\left\{\frac{1}{(1+r)^{\bar{t}+1}-(1+r)}\right\}\xi(t^*) \tag{1.26}$$

where

$$\xi(t^*) = (t^*+\bar{t}-T)(1+r)^{T-t^*} - B(t^*)\bar{t}$$

$$B(t^*) = \frac{1 - (1+r)^{T-t^*-\bar{t}}}{1 - (1+r)^{-\bar{t}}}$$

Since $\xi(T) = 0$ and by the assumption (1) $\frac{d\xi}{dt^*} < 0$, we have $\xi(t^*) > 0$ $t^* \varepsilon [1,T]$, which implies $\frac{dD_T^1}{dr} \geq 0$. (Equality holds for $t^*=T$)

(ii) By differentiating D_T^1 with respect to t^* we have

$$\frac{dD_T^1}{dt^*} = (P-\emptyset)\frac{dB}{dt^*} > 0. \tag{1.27}$$

Thus assumption (2) implies

$$t^*(r_1) > t^*(r_2) \qquad \text{for } r_1 < r_2.$$

Q.E.D.

V. CONCLUDING REMARKS

We have formulated a model in which the consumer maximizes the discounted present value of utility with respect to the consumption time path and replacement timing. We note that the derived optimal time path crucially depends upon the ratio of the interest rate to the subjective time preference, and that the time path will shift either upward or downward depending upon the magnitude of the "wealth effect" as the price of new housing changes. The ambiguity involved seems to be typical of that associated with any utility analysis.

The optimal replacement time is characterized in terms of its sensitivity to the changes in parameters such as the price of a new house and rate of interest. It is shown that one cannot unequivocally determine the direction of change in the replacement timing as the parameters change. The "wealth effect," debt constraints, and "financial position" must all be taken into account.

Even though the analysis here is confined to the situation where replacement takes place once at most, the model can be generalized to handle n replacements $(n = o,1,\ldots,T)$.[14]

FOOTNOTES

1. Following the pioneer work of J.S. Taylor [7] and Harold Hotelling [5], Gabriel Preinreich [6] introduces the important concept of the replacement chain, which shows that the economic replacement of a machine is affected by the entire chain of successive renewals over the firm's planning horizon. Armen Alchian [1] considers the replacement problem in the explicit functional framework and shows the simple algorithm of the solution. Dynamic programming technique is successfully applied by Richard Bellman [2], and further extended by Stuart Dreyfus [3]. Dreyfus presents a general solution to the problem of evaluating the decision to keep or to replace involving a finite, fixed time horizon model.

2. Although some studies have been made concerning the optimal life of consumer durables (A.H. Fox [4], A.A. Alchian [1]. Unfortunately those studies are not dealt with in the more appropriate framework of inter-temporal utility maximization.

3. The relevance of such study to the demand for consumer durables has been neglected in the past. However such study will certainly shed some light on the hitherto obscure relationship between the components of the demand for durables.

4. In contrast to the firm and organization, probably it is more appropriate to assume a finite, fixed time horizon for the consumer theory. Expected remaining life length may constitute the upper bound for the horizon.

5. That is, in determining the optimal replacement time, t^*, if it turns out that $t^* > T$, the end of his finite horizon, he will keep his present home.

6. If the consumer owns a house completely then $\bar{d}$ and the initial outstanding debt, are both equal to zero.

7. Since $D_t^0 = (1+r)\, D_{t-1}^0 - \bar{d}$.

8. We assume that the date $(t^*+\bar{t})$ when the consumer completes his debt paument is farther than the time horizon T. Note $\bar{t}$ is given by

$$(1+r)^{\bar{t}}\, \{P-\emptyset(P)\} - f(P) \sum_{t=t^*+1}^{t^*+\bar{t}} (1+r)^{t-t^*-1} = 0.$$

Thus $f(P)$ can be expressed by the following

$$f(P) = \{P-\phi(P)\}\{\frac{r}{1-(1+r)^{-t}}\}.$$

Substitution of the above into (1.4b) yields (1.4b').

9. We will only consider the case where $C_t > 0$ ∀t. This is possible by assumption (1.8b) below

$$(\partial/\partial C_t)U(0,h_t) = \infty .$$

10. For this reason one can derive the characterization by means of control theory when the problem is set up in continuous form. In fact he can derive exactly the same result. However, for the continuous case, it is not possible to perform the step B, which is only possible for the discrete and finite time period problems.

11. It can be shown that if the market value of the new house does not depreciate substantially and if the market value of the old house is more sensitive to the change in the price of a new house, then the final stock of wealth may not fall as the price of a new house rises, i.e. if we define "wealth effect" to be the sum of dH^0_{t*}/dP and $\gamma^1(\cdot)h^1_T$ in the equation below the presence of a strong "wealth effect" would lead to

$$\frac{dW_T}{dP} \geq 0 \text{ since } \left.\frac{dW_T}{dP}\right|_{t*=\text{const}} = (1+r)^{T-t*}\{\frac{dH^0_{t*}}{dP} - \phi'(P)\} + \gamma^1(\cdot)h^1_T - \{1-\phi'(P)\}B(t*).$$

12. We are grateful to the referee for pointing out an error in the original construction of Corollary 3.

13. The exact conditions to ensure $(dW_T/dr) \leq 0$ are unknown; however, the situation where

$$B \geq \sum_{t=1}^{T} (1+r)^{T-t} \frac{dC_t}{dr}$$

which is equivalent to

$$\frac{dW_T}{dr} \leq 0$$

occurs is certainly conceivable.

14. This would also cover the case in which the lifetime number and expected dates of housing purchases were the object of the consumers decision process. However, the present model can certainly be considered as a "lifetime" model for those prospective buyers say over 50, and the complexity of the general case prohibits its exposition here.

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